INTERDISCIPLINARY LIVELY APPLICATIONS PROJECT

Developments in Mathematics

PARALLEL DEVELOPMENTS IN PHILOSOPHY AND MATHEMATICS IN INDIA

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Contents

1. Introduction

This project explores some of the parallel developments in philosophy and mathematics in South Asia up to about the year 1500. It was created by the authors for a team-taught class on the development of philosophy and mathematics in India. It has also been used by the first author in first-semester calculus to show how the idea of the derivative evolved in India from the study of difference equations, totally independent of the geometric idea of tangent lines.

It should not be surprising that advances in both philosophy and mathematics were often made at the same times and in the same places. Both philosophy and mathematics flourish in times of stability and prosperity, and both are fruits of the intellectual struggle to make sense of our world.

Philosophy and religion are closely related. Both deal with metaphysical reality. Perhaps surprising to the contemporary reader is that mathematics and religion have also been closely related throughout history in many cultures. Even in Western Europe where science and religion are often considered antithetical, the Vatican has maintained an astronomical observatory that was founded in the sixteenth century and is still today an important center for scientific research.

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In South Asia, sophisticated mathematics grew from the attempt to predict celestial phenomena. It was the priests who had the time and the intellectual training that enabled them to pursue this study. And the priests had an incentive: To understand the workings of the heavens is to come closer to understanding the nature of transcendent reality.

The mathematics in this project will focus on the development of trigonometry. Trigonometry arose from and for over fifteen hundred years was used exclusively for the study of astronomy and astrology. Basic concepts of trigonometry were developed in classical Greece. These were imported to India where they flourished, creating the modern subject we know today. Hindu and Jain priests went so far as to discover and use the basic calculus of trigonometric functions. Always, the purpose was to further investigations of astronomical phenomena. Again, this is not peculiar to South Asia. In Western Europe throughout the Middle Ages and Renaissance, the terms Mathematician, Astronomer, and Astrologer were considered synonomous.

2. Greek origins of trigonometry

Trigonometry first appeared in the eastern Mediterranean. Hipparchus of Nicæa in what is now Turkey (c. 161–126 BCE) is considered the originator of trigonometry. He developed it to determine and predict positions of planets. The first problem to which he applied this new tool was the analysis of a very disturbing discovery: The universe is lop-sided.

To explain his work, we need to begin with the Greek understanding of the universe. This starts with the assumption that the earth is stationary. While this was debated in early Greek science—does the earth go around the sun or the sun around the earth?—the simple fact that we perceive no sense of motion is a powerful indication that the earth does not move. In fact, when in the early seventeenth century it became clear that the earth revolves about the sun, it created a tremendous problem for scientists: How to explain how this was possible? How could it be that we were spinning at thousands of miles per hour and hurtling through space at even greater speeds without experiencing any of this? Surely if the earth did move, we would have been flung off long ago. Newton's great accomplishment in his *Mathematical Principles of Natural Philosophy* was to solve this problem.

The astronomy of Hipparchus begins with a fixed and immovable earth. Above it is the great dome of the night sky, rotating once in every 24 hours. In far antiquity it was realized that the stars do not actually disappear during the day. They are present, but impossible to see against the glare of the sun. The position of the sun in this dome is not fixed. During the year, it travels in its own circle, called the *ecliptic*, through the constellations. One can tell the season by locating the position of the sun in its annual journey around this great circle. This is what the zodiac does. The sign of the zodiac describes the location of the sun by pinpointing the constellation in which it is located.

Most stars are fixed in the rotating dome of the sky, but a few, called the *wanderers* or, in Greek, the *planetes* (hence our word planets), also move across the dome following this same ecliptic circle. If the position of the sun is so important in determining seasons of heat and cold, rain and drought, it appears self-evident that

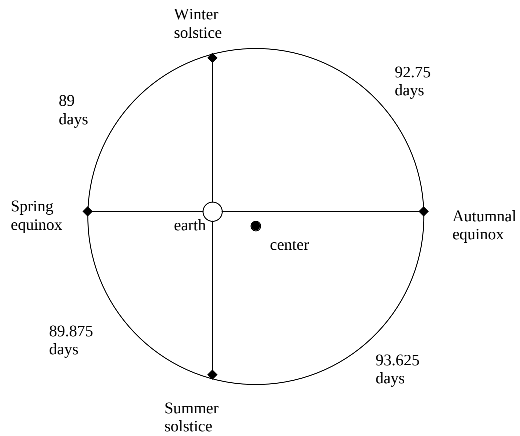

FIGURE 1. The unequal seasons, rounded to nearest 1/8 day.

the positions of the wanderers should have important—if more subtle—influences on our lives. Astronomy/astrology was born.

The philosophers of ancient Greece had developed an orderly world-view that put the earth in the center of the universe with the moon, sun, and planets embedded in concentric, ethereal spheres that rotated with perfect regularity around it. This model became the basis for a comprehensive understanding of the universe that was tight and consistent and would last for almost two millennia. Hipparchus was one of the first to find a serious flaw in this system.

The four cardinal points of the great circle traveled by the sun mark the boundaries of the seasons: winter solstice, spring equinox, summer solstice, and autumnal equinox. If the sun travels the ecliptic at constant speed, then this model would imply that the four seasons should be of equal length. They are not (see Figure 1). Winter solstice to spring equinox is a short 89 days. Spring is almost 90 days. Summer, the longest season, is over $93\,{}^1\!/\!_2$ days. And fall comes close to 93 days. If, in fact, the sun moves at a constant speed, this can only mean that the earth is off-center. Hipparchus tackled the problem of calculating the position of the earth.

Question 1. *Given the lengths of the seasons, how far off center is the earth?*

Hint 1a The first thing that Hipparchus did was to determine the arclength traveled by the sun during each season. Hipparchus measured this arclength in degrees. Today we use degrees to measure angles, but in Greece and in India and even in western Europe until the eighteenth century, degrees were used to measure distance along the circumference of a circle. The total circumference is 360°. Winter lasts 89 days which is 89/365.25 year. How far, in degrees, does the sun travel during winter? How far does the sun travel in each of the other seasons?

Hint 1b If the circumference of this circle is 360, what is its radius? This will be the value of R.

Hint 1c We will need to find the length of the chord that connects the winter and summer solstice and the length of the chord that connects the spring and autumnal equinox. Before we do that, consider how we will use this information. Show that if a chord has length L, measured in degrees, and the radius of the circle is R, then

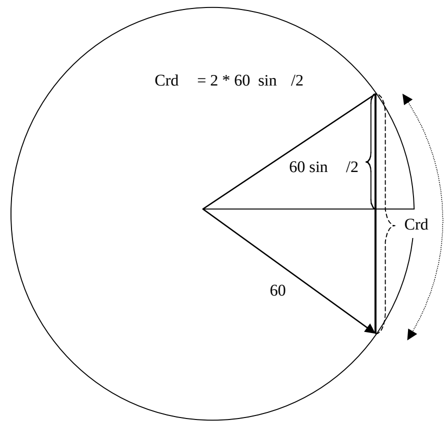

FIGURE 2. The relationship between crd α and $\sin\alpha$.

we can use the Pythagorean Theorem to find the distance from the center of the circle to this chord:

$$\text{distance} \;=\; \sqrt{R^2 - (L/2)^2}.$$

Hint 1d The crux of the problem is to find the lengths of the chords. Given arclength α on a circle of radius R, we denote the length of the chord that connects the endpoints of the arc by crd α. Using Figure 2, show that crd $\alpha = 2R\sin(\alpha/2)$ where we measure the argument of the sine in degrees. Set $R = 360/2\pi$ and use your calculator (remember to put it in degree mode) to find the lengths of the two chords. Now find the distance from the earth to the center of the universe (measured in degrees).

For the last part of this question, you had to use a calculator to find the value of the sine. Hipparchus didn't have a calculator. In fact, he had no idea what a sine was. (Remember, he is the father of trigonometry. Sines and cosines did not exist. He had to invent them.) His first task was to construct a table of the values he would need. The table he constructed gave the values of the chords for different arclengths. From the relationship given in Hint 1d, you should be able to turn a table of sines into a table of chords, or a table of chords into a table of sines.

In the second century CE, Ptolemy of Alexandria (in Egypt) wrote one of the greatest astronomical works of all times, the *Almagest*. (This is the Arabic name for this work. It means "The Greatest.") He gave a table of chords for a circle of radius $R = 60$, part of which is reproduced below. The lengths of the chords are given in base 60: 10/27/32 means 10 + 27/60 + 32/3600.

Arc α	crd α
$10°$	10/27/32
$10\frac{1}{2}°$	10/58/49
$11°$	11/30/05
$11\frac{1}{2}°$	12/01/21
$12°$	12/32/36
$12\frac{1}{2}°$	13/03/50
$13°$	13/35/04
$13\frac{1}{2}°$	14/06/16
$14°$	14/37/27
$14\frac{1}{2}°$	15/08/38
$15°$	15/39/47

Question 2. *Convert Ptolemy's table into a table of sines in decimal notation, and then check his accuracy.*

He constructed his table starting with the knowledge that crd $60° = R$ and crd $72° = R\sqrt{(5-\sqrt{5})/2}$. Combining these with the half-arc formula (expressing crd $(\alpha/2)$ in terms of R and crd α), he could calculate crd $30°$, crd $15°$, crd $7\frac{1}{2}°$, crd $36°$, crd $18°$, and crd $9°$. Using the formula for crd $(\alpha-\beta)$ in terms of R, crd α, and crd β, he found crd $1\frac{1}{2}°$. He used the approximation for small arcs,

$$\frac{\text{crd }\alpha}{\text{crd }\beta} \approx \frac{\alpha}{\beta},$$

to approximate crd $1°$. Using the half-arc formula again, he could find crd $\frac{1}{2}°$. He could fill in the missing values by using the sum of arcs formula.

3. Laying the intellectual groundwork in South Asia

3.1. **Philosophy in the** *Vedas***.** Throughout the middle of the second millennium BCE, a nomadic people migrated into northern India from neighboring Iran. Initially settling along the upper reaches of the Indus Valley, they soon came to dominate much of northern India. When these nomads entered India it was populated by ethnically and linguistically diverse groups of people, but the Indus Valley itself was home to a sophisticated (but probably already declining) civilization that has come to be known as the Indus Valley Civilization.

The entering nomads, known as Āryans (derived from *ārya* which means noble), left behind a vast body of literature composed between the end of the second millennium BCE and the middle of the first millennium BCE. Known as the *Vedas*, these texts are still recognized by contemporary Hindus as having the highest scriptural authority. The language of the *Vedas*, a close precursor of classical Sanskrit, belongs to the same linguistic family as Greek, Latin and many modern European languages. The Āryans who filtered into India must therefore have originated from the same population that had migrated westwards into Europe. Once in India, the Āryans were influenced by the indigenous cultures that they encountered. One of the enduring and most elusive questions about Vedic civilization is to ascertain the extent of this influence and to separate it from ideas brought by the Āryans themselves into India.

Vedic practice and thought centered on the performance of a complex set of ritual sacrifices, performed on behalf of wealthy and powerful patrons by hereditary groups of ritual specialists, the *brāhmanas* (henceforth written as "brahmin"). Vedic sacrifices were performed on a daily basis, at crucial times during the year such as the juncture between seasons, and on exceptional occasions such as the assertion of sovereignty by a conquering king. Although the gods were believed to partake of these ritual offerings, the efficacy of the ritual depended on its correct performance, not on the capricious will of the gods. The underlying rationale of these sacrificial rituals suggests a belief in hidden correspondences between ritual and cosmic realities. As bearers of this ritual knowledge, the *brahmins* held a powerful position in society alongside the ruling elites, or *kṣatriyas*, on whose behalf they performed the sacrifices. The *brahmins* preserved this knowledge from one generation to the next through the establishment of priestly lineages responsible for memorizing and transmitting some portion of this sacrificial knowledge. Consisting of hymns to the Vedic gods and sacrificial formulæ, the earliest sections of the four books of the *Vedas* (the four books are the *Ṛg, Sāma, Yajur* and *Atharva Veda*) began as orally transmitted teachings and did not become written manuscripts until the beginning of the Common Era.

The belief in a knowable, orderly cosmos amenable to human manipulation was an underlying theme of Vedic thought. It remains an important theme for subsequent Indian philosophical thought. Already in the *Vedas* this idea of order had been extended to human conduct, and moral virtue was understood as acting in accordance with the cosmic laws. Society itself was ordered into four hereditary groups known as *varṇas*, each one with its appropriate and specialized sphere of action. One Vedic hymn describes creation in terms of the sacrifice of a great primal being. From the sacrifice of his thousand headed body came the whole of the created order, including the four *varṇas*: the *brahmins* (priests) came from his mouth, the *kṣatriyas* (warriors) from his arms, the *vaiśyas* (artisans) from his thighs, and the *śūdras* (servants) from his feet. What this hymn tells us is that these social divisions were viewed as much a part of the natural order of things as the division between the sun and the moon, or the sheep and the goat.

The Vedic sacrificial rituals were the rituals of a nomadic people and involved the construction of often elaborate but temporary altars. Like the rituals of the ancient Greeks, the construction of these altars necessitated precise changes to be made in the shape of the bricks used in their construction. A knowledge of geometry and astronomy was required to ensure the correct construction of these altars, in the correct location, at the correct time. This knowledge was recorded in appendices to the *Vedas* where we can find, for example, a statement of what has come to be known as the theorem of Pythagoras. Scholars continue to debate the origins of this knowledge. Some speculate that knowledge of the Pythagorean theorem spread into Greece and India from common Indo-European origins.

3.2. **Philosophy in the *Upaniṣads*.** In 800 BCE India there was no Buddhism, no Jainism, and little of what we would now characterize as being typically Hindu. Many of the ideas associated with the Indian religious and philosophical traditions were formed during this period (c.700–200 BCE), a time of both great social change and intellectual speculation. Almost a thousand years after the demise of the cities of the Indus Valley Civilization, new urban centers appeared further eastward in the Ganges valley. In this part of India, small tribal societies were being consolidated into larger kingdoms and new cities appeared that were to become the administrative and military centers for these larger political entities. This more cosmopolitan environment facilitated the exchange of intellectual ideas as well as material goods, and undoubtedly this social upheaval resulted in the growth of intellectual speculation that marked this period. Such speculations are recorded in the *Upaniṣads* (pronounced "Upanishads"), texts that constitute the final portion of the *Vedas*. In the *Upaniṣads*, we see the formation of a worldview that became the framework for almost all subsequent Indian philosophical and religious thought.

This worldview can be grasped by understanding three key concepts: *saṃsāra*, *karma* and *mokṣa*. *Saṃsāra* refers to the idea that each person lives not only once but through a series of lives. It involves a cyclical view of existence inasmuch as each death is followed by rebirth, either in this world or elsewhere. As belief in *saṃsāra* became more clearly articulated, the Vedic belief in an orderly cosmos persisted in the form of causal regularities that link one life to the next. *Karma*, originally referring only to ritual actions, became a more generic term, referring to all actions of moral consequence. An individual's actions, whether good or bad, generate the energy that drives the process of rebirth, and the actions of a past life have predictable causal repercussions in a subsequent life. Simply understood, good actions generate a good rebirth, and bad actions a bad. "Good" and "bad" must be viewed as relative terms, however, because the idea of any kind of continued worldly existence came to be seen as problematic. Although the robust worldliness of the earlier Vedic hymns was present in the *Upaniṣads*, temporal existence was increasingly seen as unsatisfactory and inevitably involved suffering. The search for a way to transcend or be liberated (*mokṣa*) from an existence determined by the workings of *karma* becomes one of the central questions of the *Upaniṣads*.

One of the most prominent of teachers depicted in the *Bṛhadāraṇyaka Upaniṣad*, Yājñavalkya, under repeated questioning at a great public debate, reduced the number of gods from "three and three hundred, and three and three thousand" (3,306) gods to one and a half, and finally to one god. Apparently, Yājñavalkya taught that beneath the multiplicity of the Vedic pantheon is a simpler underlying transcendental reality. After much speculation, this reality was identified as *brahman*. *Brahman* had originally referred to the sacred power associated with the chants or *mantras* used in the Vedic sacrifices. In the *Upaniṣads*, the concept of *brahman* grew until it was firmly established as a fundamental principle pervading and sustaining all of the created worlds. The *Vedas* had presupposed a linkage between cosmic and ritual reality. In the *Upaniṣads*, this logic was extended to a belief in a correspondence between human and cosmic reality.

The *Upaniṣads* taught that *brahman* is present in each individual as the Self (*ātman*) and liberation from the world of transmigration is effected through an experiential awareness of one's essential unity with the supra-mundane reality of *brahman*. When *brahman* was considered only in ritualistic terms, then ritual knowledge gave a sufficient knowledge of its reality. In the *Upaniṣads*, however, there was a movement toward the authority of experience. *Brahman* became conceptualized in such a way that it was to be known on the basis of experience.

There was a shift toward what we today would characterize as psychology. The identification of *brahman* and *ātman* led to the investigation of human experience in order to describe this *brahman*/*ātman* in empirical terms. For many Upaniṣadic thinkers, the relationship of the created individual to *brahman* was arrived at through an investigation of human states of consciousness waking, dreaming and deep sleep. Although impressed by the freedom and creative possibilities of dreaming, and the lack of ego and suffering of deep sleep, the Upaniṣadic thinkers eventually reached the conclusion that nothing in the domain of ordinary human experience qualifies as that state that is equivalent to *brahman*. They postulated the existence of an entirely different and transcendent state, a fourth (*turīya*) state of consciousness. This mystical fourth state goes beyond anything a normal individual would ordinarily experience. Its existence established the rationale for the development of ascetic and yogic practices. In order to achieve this liberating state of consciousness, the individual would need to engage in a set of practices developed specifically for that end. The elitism of the Vedic sacrificial tradition was replaced by a new elite of skilled yogic practitioners.

Thus, although Upaniṣadic thought was wedded to the logic of sacrificial thought, there was a change in the character of Upaniṣadic teachings. The shift towards a new kind of experiential knowledge was paralleled by a change in recognized sources of authority. *Brahmins* were still prominent in the *Upaniṣads* as teachers, but their authority rested as much on their acquisition of this new knowledge as it did on their fluency in sacrificial ritual. These texts captured a mood of existential urgency, and we see women and non-*brahmins* engaged in this quest for *mokṣa* alongside the *brahmins*. The *Upaniṣads* reflect a society in transition.

The life of the Buddha embodied all of these changes. He was born in Kapilavastu situated at the northern edge of the Ganges plain that was by then the center of North Indian civilization. He was born among the *Śākya* people and was probably part of the ruling elite. He was not, however, a *brahmin*. Traditional accounts of his life depict him as renouncing his comfortable existence as a young man in order to seek liberation. As a renunciant he must have studied with teachers much like those depicted in the *Upaniṣads*. He became well versed in the teachings and techniques of these new teachers but ultimately challenged and rejected their solution. He rejected the metaphysical idea of *brahman* and hence its human equivalent, *ātman*. His rejection of the existence of *ātman* became a central tenet of Indian Buddhism. In his rejection of *ātman*, the Buddha was rejecting the idea that human suffering could be transcended on the basis of grasping some *one* thing, some aspect of human existence that wasn't subject to decay or suffering. Rather, his teachings embrace impermanence as a necessary quality of human existence. The Buddha diagnosed desire as the cause of both human suffering and rebirth. Buddhism is essentially a set of views and practices that aim to eliminate desire, and Buddhist enlightenment (*nirvāṇa*) is simply viewed as the extinguishing of all such desires. The Buddha rejected the belief in *ātman* precisely because understanding our impermanence was seen as a crucial step in the elimination of desire. Much more than the *Upaniṣads*, the teachings of the Buddha represented a real break from Vedic sacrificial thought.

3.3. **The Development of Logic in India.** According to Indian tradition, a fundamental purpose of philosophy is to guide the individual toward *mokṣa*, or release from the sufferings of the world. Certainly this goal was one of the principle concerns of the *Upaniṣads*. Subsequent philosophers in India continued to develop their ideas under this rubric. But a preoccupation with *mokṣa* did not exclude the development of a more analytical philosophy. For many thinkers, holding correct views about the world became a necessary component of *mokṣa*, and the development of such views required the critical skills of logical thinking.

If one understands philosophy as being a discipline that concerns itself with the construction of arguments to support particular views, then the *Upaniṣads* would not qualify as philosophical texts. They do not present critical analyses or arguments in support of the ideas offered. One can locate in the *Upaniṣads*, however, an intellectual climate conducive to the development of philosophical thought. Discussions took place either in the context of grand public debates, with valuable prizes of cattle awarded to the winner, or in the more intimate setting of small groups gathered together in the forest around a teacher or guru.

In the accounts of the life of the Buddha, we see that dialogue remained the basis for philosophical speculation, in this case dialogues between the Buddha and his followers. These texts were transmitted orally until the beginnings of the Common Era and were subject to sectarian embellishments. For this latter reason, rather than the former, no ungarnished records of the words of the Buddha exist. Nevertheless, these texts seem more realistically conversational than the *Upaniṣads*, and we can see in them an important feature of the Buddha's view of philosophical speculation. For the Buddha, "right view" was an important element in the path towards liberating the individual from desire (and hence suffering). Philosophical knowledge could have a therapeutic value, and philosophical speculation was valued insofar as it contributed to this therapeutic goal. Idle speculation, however, was regarded as a distraction. This became an important value in both the Buddhist and non-Buddhist philosophical literature of India. In the classical Hindu tradition, for example, it became incumbent on a philosopher to state at the beginning of his text what his purpose (*prayojana*) was in writing the text and to demonstrate that real doubt existed about the topic under consideration. The Buddha was also portrayed in his discourses as being very sensitive to his audience. What he taught on any particular occasion took into account the ability of his listeners to grasp what he was saying. This idea that knowledge is not an abstract, fixed entity became an important feature of Indian philosophical thought. If we understand knowledge to be a relationship between a subject and an object, then it is fair to say that from its very inception the Indian philosophical tradition was sensitive to the subjective side of this relationship.

In the centuries after the death of the Buddha, the Buddhist and Upaniṣadic traditions began to develop more sharply articulated philosophical positions. Logical thought emerged as a recognized branch of knowledge. By the third or second centuries BCE, monks and priests were trained in the practice of formal debate. Those who exhibited philosophical prowess by winning debates were powerful and revered figures. The *Milindapañha* (Questions of Milinda), a Buddhist text from about the first century BCE depicts a question and answer session between a Buddhist monk, Nāgasena, and the Greco-Bactrian King, Milinda (Milinda is the Indian form of the name Menander, who was supposedly ruler of the Indus Valley region around 150 BCE). In this dialogue, Nāgasena converted Milinda to Buddhism, and the text claims that five hundred Greeks and eighty thousand Buddhist monks were witness to their philosophical debate, no doubt an exaggeration but still indicative of the prestige of such occasions. Nāgasena specified that he would only debate the king if the debate was a debate for the wise, not a debate for the king. Nāgasena distinguishes between a debate for the king in which deference is shown to the king in virtue of his position rather than his intellectual skills, and a debate for the wise which is conducted between scholars according to rules that lay out conditions for victory and defeat. This reveals that by this time debate had become a formal activity where victory was to be decided according to well-defined rules of argument rather than on the basis of personal power.

By the beginning of the Common Era, debate manuals (*vāda-śāstra*) were widely available. These manuals taught practitioners how to win debates. They were practical manuals that recognized that debates could be won or lost for all kinds of reasons: an opponent who is strong or weak, an audience that is friendly or hostile. The manuals recognized the difference between debates where the goal was to defeat a hostile opponent and those debates between friends where both sides wished to establish the correct view. Trickery was acceptable for the first type of debate but not for the second. As in classical Greece, the study of debate provided a fertile environment for the development of a formal analysis of correct inference. This began with a separation between practical tips for winning a debate and an analysis of the correct form that a truth-establishing argument must have. Such a separation occurred in a text known as the *Nyāya Sūtras*, the foundational text for the *Nyāya* or Logic School of philosophy. The *Nyāya Sūtras* outline five steps that are necessary for the construction of a sound inference. These are as follows:

: STEP ONE: There is fire on the hill (Statement *pratijñā*)
: STEP TWO: For there is smoke (Reason *hetu*)
: STEP THREE: Wherever there is smoke there is fire, as in the kitchen (Example *udāharaṇa*)
: STEP FOUR: This is such a case (Application *upanaya*)
: STEP FIVE: Therefore there is fire on the hill (Conclusion *nigamana*)

A student of western logic is struck by the similarity of this inference pattern to western forms of reasoning, yet will also notice differences. Like mathematics, logic is often thought to be a context-transcendent form of human thought, that "truth is truth and nonsense nonsense, whether stated in Sanskrit, Tibetan or English."[1] Certainly we are able to recognize the rationality of the *Nyāya* pattern of inference despite a separation of culture and time. We are also able to see striking similarities to Aristotelean logic, so much so that some scholars have attempted to locate the source of Indian logic in classical Greece, a thesis that has no evidential support. There was a great deal of commercial, military and intellectual encounter between India and the Classical world at this time. In some areas we can readily discern the results of such encounter, as in the classically inspired Buddha sculptures that appeared at the beginning of the Common Era in the Gandharan region of northern India. In the case of Indian logic, however, its development was driven by exigencies internal to the tradition itself. Both logic and mathematics are used to solve practical problems, hence we would expect to find similarities, even in the absence of direct cultural transmission. It is also important to see the ways in which such patterns of thought are context sensitive. The philosophical context in India resulted in differences between Greek and Indian logic. The interest lies as much in locating and understanding these differences as it does in discovering global similarities.

At first glance, the *Nyāya* syllogism would strike the western-trained logician as containing redundant steps. It seems that it could easily be condensed into the

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[1]Griffiths, Paul (1982), "Notes Towards a Critique of Buddhist Karma Theory", *Religious Studies* **3** p. 278

three steps of the following syllogism without sacrificing anything necessary for the validity of the inference:

: STEP ONE: Wherever there is smoke, there is fire
: STEP TWO: There is smoke on the hill
: STEP THREE: Therefore there is fire on the hill

There are several reasons why this reduction did not happen but we can single out two as being particularly important. One reason for the greater length of the *Nyāya* syllogism is rooted in the fact that the inference was developed to be persuasive to an audience. Indian philosophy is nearly always structured as a dialogue in which ideas are developed with a particular opponent in mind. Philosophers in India distinguish between inference that one makes for one's own satisfaction and those inferences constructed for the persuasion of a third party (*parārthānumāna* -a demonstration to others). The five steps of the *Nyāya* syllogism are designed to lead another person through a thought process that will convince him of the conclusion. Secondly, nearly all Indian philosophers accept inference (*anumāna*) as a means to knowledge (*pramāṇa*). Along with perception (*pratyakṣa*), inference is our main source of knowledge about the world around us (the orthodox schools also recognized verbal testimony or *śabda* as a means of knowledge, in accordance with their acceptance of the authority of the *Vedas*). A well-constructed inference is meant to yield a piece of knowledge. Given that this was the case, there could not be the separation between soundness (a valid argument with all true premises) and validity that we see in contemporary western logic. Students of western logic must quickly learn to distinguish between the concepts of validity and soundness and learn that valid arguments will often lead to false conclusions. Only with the assurance that the premises are true can we be assured that the conclusion will also be true. If we take the above inference and rewrite it as:

: STEP ONE: Wherever there is fire, there is smoke
: STEP TWO: There is fire on the hill
: STEP THREE: Therefore there is smoke on the hill

we may still have an inference that is valid but if the premise expressed in step one is indeed false then the conclusion will be false. (For the sake of argument we are presuming, as the Indian logicians did, that smoke is always indicative of fire.) A valid argument is not acceptable as a *pramāṇa* because it will often yield a false conclusion rather than something true of the world. The Indian logicians, because inference was one of the accepted *pramāṇas*, sought an inferential form that would always yield a conclusion that was true.

Lurking behind many a sound deductive syllogism is a piece of inductive reasoning that has already established the truth of the universal premise. In our example, we would need induction to establish the truth of the concomitance between smoke and fire. Because the Indian logician needed to ensure that the conclusion of his inferences would always be true, he needed to incorporate sound inductive principles into his inference model. He needed to articulate as part of the syllogism those principles whereby we are entitled to state a concomitance between two properties, such as smoke and fire. This explains the inclusion of the example in the *Nyāya* inference, that smoke is present in a similar instance where there is fire in the kitchen. Subsequent work in Indian logic focuses on this part of the inference, examining the conditions that entitle us to state a concomitance between two properties. The Buddhist logician, Dignāga (c.400-480 CE) is particularly noted for his contributions to logic in this area. He developed three conditions that an adequate sign (*hetu*) must fulfill:

1. It must be present in the case being considered.
2. It must be present in a similar case.
3. It must not be present in a dissimilar case.

Thus if smoke is to be regarded as an adequate sign for fire, it must be present in the case under consideration (the hill), it must be present in the kitchen where we know that it exists along with fire, it must not be present in any case where we know that there is no fire. Dignāga's work in this area laid the foundations for centuries of work in the area of understanding the logical relationships between properties of things.

As in other philosophical traditions, the Indian philosophical tradition had its skeptics, most notably the Buddhist philosopher Nāgārjuna (c.100 CE). Nāgārjuna is held in the highest esteem by some Buddhist schools where he holds the title of second Buddha', and his brand of philosophical skepticism continues to fascinate contemporary commentators. Buddhist thought since the time of the Buddha has always regarded philosophical theorizing with some suspicion. The Buddha had developed a list of speculative questions that were "not fit to be asked" because they would lead the student into a morass of philosophical confusion and because the questions themselves were symptomatic of the sorts of cravings Buddhism seeks to eradicate. An example of an unfit question would be, "Are self and world eternal?" The Buddhist is suspicious of this type of question because to entertain it is to affirm the reality of such fictitious and misleading entities as "self," presupposed by the referential nature of the sentence. Nāgārjuna's whole philosophical project was based on the development of this kind of skepticism. He was very critical of the *pramāṇa* theory, attacking the epistemological foundations of the theory. Nāgārjuna held that there could be no secure foundations for knowledge and he focused his attack on the status of the *pramāṇas* themselves. What justifies our confidence in the infallibility of these *pramāṇas*? Does a *pramāṇa* need another *pramāṇa* as its foundation? Nāgārjuna argued that either we will be led into an infinite regress, or the *pramāṇa* theory itself is wrong in believing that every piece of knowledge must be reached on the basis of these *pramāṇas*.

Nāgārjuna's work belongs to a broader complex of ideas that were gaining momentum in the Buddhist tradition at this time. The canonical texts were supplemented with new sources of textual authority known as *sūtra* literature. These texts claimed their authority on the basis of changing beliefs about the nature of the historical Buddha. The historical Buddha himself became divinized to a greater degree and became part of an extensive pantheon of divine Buddha and Bodhisattva figures who were able to inspire the words of human thinkers. (The distinction between a Buddha and Bodhisattva is not clear. Initially, Bodhisattvas delayed their own final enlightenment in order to work for the enlightenment of all beings, but this distinction became blurred.) The Eightfold Path taught by the Buddha was supplanted by the Bodhisattva Path where the emphasis was on the practice of compassion, based on non-recognition of the distinction between self and other. As a result of these changes, Buddhism became a more devotional tradition. For the first time, images of the Buddha and Bodhisattva figures were produced, images that were made to be objects of veneration. Parallel changes also occured in Hinduism. The deities *Śiva* and *Viṣṇu* become the focus of well-developed mythologies and inspired major devotional movements in Hinduism.

The rise of a new Gupta dynasty reunited northern India and initiated a period of artistic and intellectual activity. Kalidāsa, one of the great poet-playwrights of India was active in the court of Candra Gupta II (c.375-415 CE). The Guptan period was also a great age of temple building, sculpture and cave painting. The new devotionalism appearing in Hinduism and Buddhism inspired new rituals. The temple was the house of a god, and his image was housed in the temple. When Fa-hsien, a Chinese Buddhist pilgrim visited Pataliputra (today known as Patna), Candra Gupta's capital, he found an affluent city where individuals enjoyed personal freedom and prosperity, a town that provided free hospitals to take care of the poor and the sick.

Astronomical works from greek Alexandria found their way to India at this time. They were either translated into Sanskrit or used as the basis for Sanskrit treatises. The foundations had been laid for a golden era of mathematics. One of the mathematicians who ushered in this age was Āryabhata. He was born in Kusumapura, near Pataliputra, in 476 and was active during the last years of the Guptan dynasty.

4. The Golden Age in India

The Indian astronomers made an important change to the trigonometry they had learned from the west. Their first innovation was to work with the half-chord, rather than the chord. They called this function the *ardha-jya* which literally means "half bowstring" or "half chord." Over time, this was shortened to jya or jiva. The value depends on the radius of the circle and the argument is the length of the half-arc (measured in degrees and minutes), but otherwise this looks like our sine,

$$\text{jya}\ \alpha \ = \ R\sin\alpha.$$

The cosine is the sine of the complementary angle. In Sanskrit, the jya of the complementary arc was called the *koti-jya* or *kojya*, and we will abbreviate it as *koj*:

$$(\text{jya}\ \alpha)^2 + (\text{koj}\ \alpha)^2 \ = \ R^2.$$

Question 3. *Use Figure 3 to prove the half-arc formula,*

$$2\,\text{jya}\ (\alpha/2) \ = \ \sqrt{(\text{jya}\ \alpha)^2 + (R - \text{koj}\ \alpha)^2}.$$

Use the sum and difference of angles formulas for the sine to find the formulas for jya $(\alpha+\beta)$ *and* jya $(\alpha-\beta)$ *in terms of the* jya *and* koj *of* α *and* β*.*

The Arabs and Persians learned their trigonometry from the Indians, and western Europe learned it from the Arabs of North Africa. That is why today we work with half-chords rather than full chords. Even the names we use can be traced back to the Sanskrit. The Arab word for the sine is *jiba*, taken from the Sanskrit. When Gherardo of Cremona translated this into Latin around 1150, he misread the arabic word as *jaib* which means "bosom" or "bay," and translated it as the Latin word "sinus."

Using a radius of $R = 3438$ and the half arc formula, Āryabhata found the value of the jya of $15°$, $7\frac{1}{2}°(= 7°30')$, and $3\frac{3}{4}°(= 3°45')$, and then used the sum of arc formula to complete a table of the twenty-five values of jya α when α is a multiple

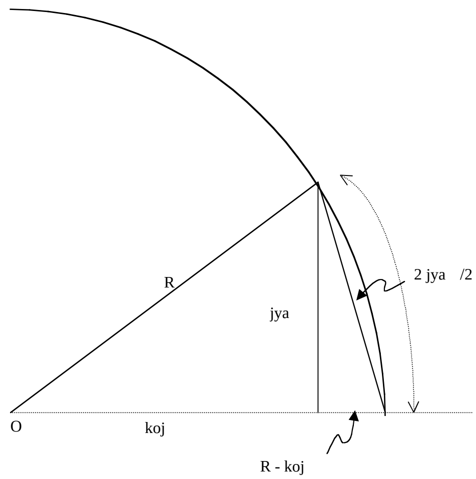

FIGURE 3. half-arc formula

of 3° 45′, $0 \leq \alpha \leq 90°$. The problem he and all other astronomers faced was how to interpolate the values in this table. Astronomical observations could be made with an accuracy of 10 or 20 minutes of arc. Linear interpolation could be used for a rough approximation, but it would not meet the needs of accuracy.

To improve the interpolated values, Āryabhata looked at the differences between the values in the table. He then calculated the differences of the differences. Part of the table that he created is given here ($\text{jya}\ x = 3438 \sin x$):

x	jya x	1st difference	2nd difference
3°45′	225		
7°30′	449	224	−2
11°15′	671	222	−3
15°	890	219	−4
18°45′	1105	215	−5
22°30′	1315	210	−5
26°15′	1520	205	−6
30°	1719	199	

Āryabhata observed that these second differences are very close to the value in the second column divided by 225, which happens to be the number of minutes in 3° 45′. He stated a general rule:

$$[\text{jya}\ (x + 225') - \text{jya}\ (x)] - [\text{jya}\ x - \text{jya}\ (x - 225')] \ \approx \ \frac{-\text{jya}\ x}{225}.$$

This was probably no more than an observation, but Āryabhata might have based his rule on the trigonometric identity

$$(1) \quad [\text{jya}\ (x + \alpha) - \text{jya}\ (x)] - [\text{jya}\ x - \text{jya}\ (x - \alpha)] \ = \ -(\text{jya}\ x) \left(\frac{2\,\text{jya}\ (\alpha/2)}{R} \right)^2.$$

Question 4. *Use the half-arc formula and the sum and difference of arcs formulas from Question 3 to prove equation (1). Show that when $R = 3438$ and $\alpha = 225'$, then $(2\,\text{jya}\ (\alpha/2)/R)^2$ is close to 1/225.*

Whether or not Āryabhata knew or used the result in equation (1), his analysis of second differences had two important consequences:

- It drew attention to the fact that if we know how to compute the second differences, we can use them to reconstruct the original function. In other words, if we know the initial values of our function and its first difference, then we can use the values of the second differences to find the values of the first differences. We then can use the initial value and the first differences to find the values of the function.
- It showed that the second difference of the sine, centered at x, is close to a constant multiple of $\sin x$. For very small arc-lengths, jya $(\alpha/2)$ is indistinguishable from $\alpha/2$. If we divide both sides of equation (1) by α^2, we get that
$$\frac{[\text{jya }(x+\alpha) - \text{jya }(x)] - [\text{jya } x - \text{jya }(x-\alpha)]}{\alpha^2} \approx \frac{-\text{jya } x}{R^2}.$$
This is not exact, but it gets closer to an exact equality as α approaches 0. Indian astronomers had begun working toward the idea of the derivative. Āryabhata cannot be credited with the discovery of the second derivative of the sine, but he did set in motion a chain of investigations that would lead to its discovery.

5. Brahmagupta and the Harsha Empire

After the collapse of Guptan rule, northern India became politically fragmented. In 606 CE the sixteen-year-old Harsha Vardhana assumed power, coming to the throne of Thaneswar, north of Delhi. During the forty-one years of his reign he extended his rule so that northern India again became united under a single power. Harsha supported the practice of Buddhism, but also the increasingly popular devotional forms of Hinduism (known as *bhakti*) gaining ascendancy in northern India. It was during this period of relative political stability that the mathematician Brahmagupta was active.

Brahmagupta, born 598, lived in Bhillamala (modern Bhinmal in Rajasthan) in the empire of Harsha. While he made many important mathematical discoveries, the most important for our purposes was the discovery of the quadratic interpolation formula.

We begin, as he did, with a table of values of jya with the arcs given as multiples of $3°45'$ which is the same as $3.75°$. Most of the time, the arclength α at which you need to evaluate jya α will not be a multiple of $3.75°$, so it becomes necessary to interpolate. For example, to find jya $10°$ with $R = 120$ (the value of R used by many Indian astronomers), we look in the table and see that jya $7.5° = 15.66314$ ($= 120 \sin 7.5°$) and jya $11.25° = 23.41084$. Since 10 is 2.5/3.75 of the way from 7.5 to 11.25, we can estimate jya $10°$ by linear interpolation, adding $(2.5/3.75)(23.41084 - 15.66314)$ to 15.66314. This implies that jya $10° \approx 20.82827$. The true rounded value of jya $10°$ is 20.83778.

Question 5. *Given an arclength y measured in degrees, let x be the largest multiple of 3.75 that is less than or equal to y, and let $t = y - x$. If we know the exact values of* jya x *and* jya $(x + 3.75°)$*, then the linearly interpolated value of* jya $y = 120 \sin y$ *(measured in degrees) is given by*
$$\text{jya } y \approx \text{jya } x + (t/3.75)(\text{jya }(x+3.75) - \text{jya } x),$$

where

$$x = 3.75 \left\lfloor \frac{y}{3.75} \right\rfloor, \quad t = y - x, \quad \text{jya } \theta = 120 \sin \theta,$$

and $\lfloor v \rfloor$ *denotes the greatest integer less than or equal to* v*. For* $0 \le y \le 90^\circ$*, plot the difference between the true value of* jya y *and the linearly interpolated value.*

Brahmagupta showed how to use both the first and second differences to improve the interpolated values. Instead of using a linear approximation, he found a quadratic approximation. The formula that he discovered is

$$\text{jya } (x+t) \approx \text{jya } x + t\left(\frac{\text{jya } (x+\alpha) - \text{jya } (x-\alpha)}{2\alpha}\right) + \frac{t^2}{2}\left(\frac{\text{jya } (x+\alpha) - 2\text{jya } x + \text{jya } (x-\alpha)}{\alpha^2}\right). \tag{2}$$

It is worth noting that this formula is valid no matter what value of R we take.

What Brahmagupta had discovered is the quadratic case of the Newton interpolation formula. The right side is the unique quadratic polynomial in t that agrees with jya $(x+t)$ when $t = -\alpha$, 0, or α.

Question 6. *Show that the quadratic polynomial in* t *on the right side of (2), is equal to* jya $(x+t)$ *when* $t = -\alpha$*,* 0*, or* α*.*

Question 7. *Using the fact that* jya $7.5^\circ = 15.66314$, jya $11.25^\circ = 23.41084$*, and* jya $15^\circ = 31.05629$*, use Brahmagupta's interpolation method to approximate* jya 10°*. Compare this with the approximation* jya $10^\circ \approx 20.82827$ *that was obtained by linear interpolation.*

Question 8. *Assume that we have a table of values of* jya x *when* x *is a multiple of* 3.75°*. Given an arc* y*, measured in degrees, let* x *be the closest multiple of* 3.75 *and let* $t = y - x$*. For* $0 \le y \le 90^\circ$*, plot the difference between* jya $y = 120 \sin y$ *(measured in degrees) and the quadratic approximation*

$$\text{jya } x + \left(\frac{t}{7.5}\right)(\text{jya } (x+3.75) - \text{jya } (x-3.75)) + \left(\frac{t^2}{28.125}\right)(\text{jya } (x+3.75) - 2\text{jya } x + \text{jya } (x-3.75)), \tag{3}$$

where

$$x = 3.75 \left\lfloor \frac{y+1.825}{3.75} \right\rfloor, \quad t = y - x, \quad \text{jya } \theta = 120 \sin \theta.$$

Compare the error functions for the linear and quadratic approximations. What do you notice?

6. Śaṃkara and *Advaita Vedānta* Philosophy

Southern India saw a period of political consolidation under the Pallava and Chola dynasties who dominated India from about 600 CE for a period of six hundred years. From that time, south India has played an increasingly important role in the development of the intellectual and artistic traditions of classical India.

Around the sixth century CE, a new figure of religious authority emerged, that of the poet-saint, practicing an intense, ecstatic form of bhakti. These poet-saints, who were followers of either *Viṣṇu* or *Śiva*, wandered over the countryside of the

Tamil region of south India composing and singing devotional poems. In contrast to the Sanskrit texts of brahmanical Hinduism, these poet-saints used the vernacular Tamil language as their medium of expression, thus establishing an immediate connection with the mass of the population not literate in Sanskrit. Many of them were non-*brahmins*, and some were women. Unlike earlier *bhakti* texts, the poems of these individuals expressed the emotional intensity of their relationship to God. This form of Hinduism spread rapidly, with poet-saints appearing in each distinct linguistic region of India, both in the north and in the south. Their poetry inspired the birth of vernacular literary traditions and gave Hinduism a popular appeal that ultimately contributed to the demise of Buddhism in India. To this day, the pilgrimage sites and poetry associated with these poet-saints contribute to the strong regional identities found in India. Pious individuals continue to venerate the poet-saints of their region, memorizing and singing their poems in the temples and on pilgrimage.

There is no single set of practices or beliefs that define what it means to be a Hindu. Some sectarian Hindus may worship *Śiva* as the Supreme God, others *Viṣṇu*, while others maintain a devotion to a plurality of Gods or understand absolute reality in terms of the impersonal *brahman*. In addition, each region has its distinctive set of practices and beliefs. The Indian philosophical tradition with its doctrines of different levels of truth was epistemologically suited to such a plurality of belief. This inclusivism is evident in the work of one of India's best known philosophers, Śaṃkara. A near contemporary of the mathematician Govindaswāmi and born in the same region of India in Kerala, Śaṃkara (c.788-820) is the leading philosopher of the *Advaita Vedānta Darśana*, one of the six orthodox schools of philsophical thought. Central to this system is an unqualified commitment to a non-dualistic (*advaita*) interpretation of the *Upaniṣads*, those texts constituting the end of the *Vedas* (*Vedānta*). For Śaṃkara, from an absolute point of view, only unqualified *brahman* has existence. The philosophical problems and questions generated by Śaṃkara's metaphysical position are legion. If, from an absolute point of view, reality is non-dual, how can Śaṃkara account for the apparent multiplicity of the world we inhabit? What is the status of such a world, and what is its relationship to the unqualified *brahman*? In his short life, Śaṃkara articulated a philosophical position that maintained an unwavering commitment to the non-dualism of absolute reality, yet gave a coherent account of the reason for and the status of the reality of ordinary human experience.

Śaṃkara's epistemology rested on the idea that our judgments about the world are influenced by our past experiences. If I mistakenly judge a coiled piece of rope lying in the corner of the room to be a snake (Śaṃkara's well-known example) then I am introducing something from my past in to my present experience. (Someone living in India, where snakes are common, would therefore be much more likely to mistakenly view the rope as a snake than someone living in England where snakes are rarely encountered.) Perception is not simply *seeing* but, rather, *seeing-as*. Śaṃkara calls this phenomenon *adhyāsa* (superimposition). Superimposition occurs when a past experience colors a present perception because of some resemblance between them. In this case, a past experience of a snake has colored the present perception of the rope because of a similarity in visual appearance. When I'm perceiving the rope as a snake then, at that moment, the snake is real for me. If I walk towards the rope to investigate further, then I will revise my initial judgment and perceive the rope as a rope. Knowledge, then, for Śaṃkara has a provisional status. We hold something to be true until a good reason comes along for us to revise our judgment.

Śaṃkara used this description of epistemological error as a way of understanding the human predicament in general. According to Śaṃkara, ordinary human experience of the world is, like the rope and snake example, a mis-perception based on ignorance (*avidyā*). In Śaṃkara's thought the world of ordinary experience is known as *māyā*. This term is often translated as meaning *illusion*, but this leads to a misconception about Śaṃkara, that he teaches that the world is an illusion. The term *māyā* is better translated as *appearance*, for this captures the status that the world of ordinary experience has for Śaṃkara. Certainly it has more reality than illusory experiences. Our judgment that there is water on the road will be displaced if, upon closer inspection, it disappears. Śaṃkara thus preserves the important practical distinction between real and illusory experience. Yet Śaṃkara also holds that what we commonly judge to be veridical experience is itself open to revision. Through a process of self-transformation, it is possible to view the reality of ordinary human experience as mistaken. The multiplicity of the world of ordinary human experience based on dualities, such as subject versus the object and pain versus pleasure, disappear when absolute reality is experienced as non-dual, without distinguishing qualities (*nirguṇa brahman*). If absolute reality consists only in unqualified *brahman*, then ontologically nothing other than *brahman* has real existence. Śaṃkara is thus committed to a theory of causation that argues for the pre-existence of the effect in the cause. This theory of the pre-existence of the effect in the cause (*satkāryavāda*) opposes both the Buddhist and realist *Nyāya* theories of causation that argue for the occurrence of real change, and the independent existence of the effect.

Śaṃkara's philosophy captures the inclusivist values of Hinduism. Perhaps this explains why his philosophy is so often identified as being quintessentially Hindu in character. Historically his thought created a cohesive force, binding together the many different ways in which it was possible to practice Hinduism. Thus Śaṃkara is able to value the passionate *bhakti* of the poet-saints as a view that moves the individual away from an ego-centered existence to the perception of ultimate reality as non-dual in nature. Although other schools might disagree with Śaṃkara about the non-dual nature of ultimate reality, his inclusivist epistemology became a widely used strategy. Thus, the poet-saint who saw the distinction between God and the individual as being ultimately real could use Śaṃkara's epistemological strategy as a way of valuing Śaṃkara's beliefs. The poet-saint could view such beliefs as being an important step on the way to realizing the absolute reality of a loving God.

7. Govindaswāmi and Keralese mathematics

Govindaswāmi lived in Kerala near Śaṃkara and practiced his mathematics near the end of or shortly after Śaṃkara's life. Govindaswāmi improved the approximation of π and elaborated on many of Brahmagupta's methods, including his method of quadratic interpolation. Later Keralese astronomers built on these interpolation methods, extending them to cubic and hgiher degree polynomials. More importantly, they discovered what today we know as the derivatives of the sine and cosine, turning the interpolating polynomials into infinite summations that could be used be used to obtain estimates of arbitrary precision.

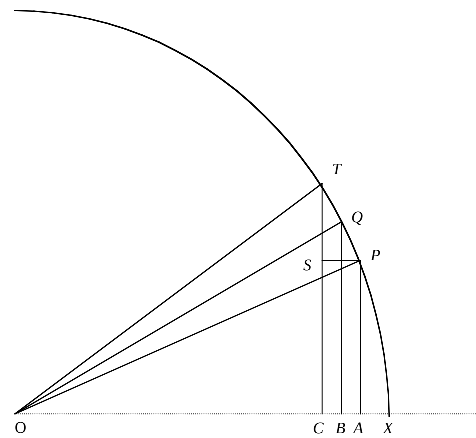

FIGURE 4. $ST = \Delta(\text{jya } x)$ and $PS = \Delta(\text{koj } x)$.

The key step in passing from the interpolating polynomials to the infinite summations is the recognition that

$$\lim_{\alpha \to 0} \frac{\text{jya } (x+\alpha) \; - \; \text{jya } x}{\alpha} = \frac{\text{koj } x}{R}, \quad \text{and} \tag{4}$$

$$\lim_{\alpha \to 0} \frac{\text{koj } (x+\alpha) \; - \; \text{koj } x}{\alpha} = \frac{-\text{jya } x}{R}. \tag{5}$$

Indian mathematicians did not use the concept of limits, but they did recognize that the fraction on the left can made as close as desired to the fraction on the right by taking α sufficiently close to 0. As we will see when we prove these results, it is important that the arclengths and radius are measured in the same units. If we measure the arclength in degrees, then the radius must be $360/2\pi$. If measured in minutes, then $R = 21600/2\pi$. If we take $R = 1$, then the arclength is measured in *radians* (multiples of the radius). If the radius is $R = 120$, then it takes 120 arclength units to equal one radian.

We don't know when these derivatives were first discovered. The earliest known proof (which we will see in a moment) was given by Jyesthadeva of Kerala in the early 1500s, but the derivatives were used by Paramśvara around 1400 for a cubic polynomial approximation, and they may have been known as ealy as the twelfth century.

In Figure 4, we let $\widehat{PX}$ be the arclength x and $\widehat{PT}$ be α, the amount by which the arclength increases. The quantity jya $(x+\alpha)$ − jya x is the amount that our function changes, and we will denote it by the special notation

$$\Delta(\text{jya } x) = \text{jya}\,(x+\alpha) - \text{jya } x.$$

The problem is to estimate $ST = \Delta(\text{jya } x)$ and $PS = -\Delta(\text{koj } \alpha)$. We mark Q, the midpoint of arc $\widehat{PT}$, and note that OQ is the perpendicular bisector of chord PT.

Question 9. *Explain why* koj $(x+\alpha) <$ koj x *for* $\alpha > 0$*, and therefore why* $PS = -\Delta(\text{koj } \alpha)$.

For a small change in arclength, the chord PT is a very good approximation to the arc $\alpha = \widehat{PT}$. (This is why chords, including the diameter, must be measured in the same units as the arclength.) The ratio α/PT can be made as close to 1 as we wish by taking α sufficiently small. Also, BQ equals jya$(x + {}^1\!/_2\alpha)$, and so the ratio $BQ/\text{jya } x$ can be brought close to 1 by taking α close to 0. Similarly, $OB/\text{koj } x$ can be taken to be arbitrarily close to 1. Using the special notation $\lim_{\alpha\to 0}$, we write these observations as

$$\lim_{\alpha\to 0} \frac{\alpha}{PT} \;=\; \lim_{\alpha\to 0} \frac{BQ}{\text{jya } x} \;=\; \lim_{\alpha\to 0} \frac{OB}{\text{koj } x} \;=\; 1.$$

Question 10. *Show that triangle TSP is similar to triangle OBQ. Hint: Let Y be the point of intersection of OQ and TC. Use the fact that angle OYC equals angle TYQ.*

From the similarity of triangles TSP and OBQ, we see that

$$\frac{ST}{PT} = \frac{OB}{OQ} \quad\Longrightarrow\quad \frac{\Delta(\text{jya } x)}{\alpha}\,\frac{\alpha}{PT} \;=\; \frac{\text{koj } x}{R}\,\frac{OB}{\text{koj } x}, \tag{6}$$

$$\frac{PS}{PT} = \frac{BQ}{OQ} \quad\Longrightarrow\quad \frac{-\Delta(\text{koj } x)}{\alpha}\,\frac{\alpha}{PT} \;=\; \frac{\text{jya } x}{R}\,\frac{BQ}{\text{jya } x}. \tag{7}$$

We drop the ratios that can be made arbitrarily close to 1, and we get the desired result.

Question 11. *Using equations (4) and (5), show that as α approaches 0, Brahmagupta's quadratic interpolation formula, equation (2), becomes*

$$\text{jya } (x+t) \;\approx\; \text{jya } x + \frac{t}{R}\,\text{koj } x - \frac{t^2}{2R^2}\,\text{jya } x. \tag{8}$$

Question 12. *Assume that we have a table of values of* jya x *when x is a multiple of* $3.75°$. *Given an arc y, measured in degrees, let x be the closest multiple of* 3.75 *and let $t = y - x$. For $0 \le y \le 90°$, plot the difference between* jya $y = (180/\pi)\sin y$ *(measured in degrees) and the quadratic approximation*

$$\text{jya } x + \frac{\pi t}{180}\,\text{koj } x - \frac{\pi^2 t^2}{64800}\,\text{jya } x$$

where

$$x = 3.75 \left\lfloor \frac{y + 1.825}{3.75} \right\rfloor, \quad t = y - x, \quad \text{jya } \theta = \frac{180}{\pi}\sin\theta, \quad \text{koj } \theta = \frac{180}{\pi}\cos\theta.$$

Compare the error function to those of the linear and quadratic interpolation formulas in questions 5 and 8. Note that for a legitimate comparison, you need to rescale the graph of the difference in this question by a factor of $2\pi/3$. What do you notice?

8. Conclusion

Jyesthadeva and his contemporaries went beyond polynomial approximations. Over a century before Newton or Leibniz, they showed how to use these ideas to find infinite summations that could be used to approximate jya x and koj x with arbitrary precision:

$$\text{jya } x \;=\; x - \frac{x^3}{6R^3} + \frac{x^5}{120R^5} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!\,R^{2n-1}} + \cdots \tag{9}$$

$$\text{koj } x \;=\; 1 - \frac{x^2}{2R^2} + \frac{x^4}{24R^4} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!\,R^{2n}} + \cdots. \tag{10}$$

Indian mathematicians had anticipated much of what today we would call calculus.

To this day, the philosophical and mathematical traditions in India continue to be very strong. Both are deeply embedded within the varied religious contexts of South Asia. They are part of the cultural legacy. It is impossible to say whether one is a necessary precursor to the other, if they both issue from a common source, or whether it was chance that caused both to spring forth at the same times and places. What is clear is that India possesses a rich intellectual tradition that is worthy of study.

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One of the most recent and best translations of the Upanisads, with an excellent introduction that links the Upanisads to their Vedic roots and places them in historical context.

Miller, Barbara Stoler. 1995. *Yoga: Discipline of Freedom*. Berkeley and Los Angeles: University of California Press.

A translation of Patanjali's Yoga Sutras, with an introduction by Barbara Stoler Miller in which she examines the Buddhist and Upanisadic roots of Patnajali's work.

9.2. **Indian Mathematics.**

Bose, D. M., S. N. Sen, and B. V. Subbarayappa. 1971. *A Concise History of Science in India*. Indian National Science Academy. Calcutta: Baptist Mission Press

A multi-volume work, one volume of which deals with Indian astronomy and mathematics. It provides a comprehensive overview of the principal results and the historical texts.

Datta, B., and A. N. Singh, revised by K. S. Shukla. 1980. Hindu Geometry. *Indian Journal of History of Science*. **15**:121–188.

———. 1983. Hindu Trigonometry, *Indian Journal of History of Science*. **18**:39–108.

These articles are somewhat dated despite Shukla's additions, and there is a persistent tendency to read more modern mathematics into the ancient texts than today's historians would be willing to acknowledge, but this is a good introduction to the range of mathematical topics that were studied by Indian astronomers and mathematicians.

Heath, T. 1981. *A History of Greek Mathematics*. New York: Dover. (Original work published 1921.)

Also dated, but a comprehensive and useful starting point for explorations of Greek Mathematics. It begins in Greek pre-history and continues until the fall of Byzantium, but the emphasis is on the classical period from Thales (624–527 BCE) through Diophantus (*fl. c.* 250 CE).

Katz, V. J. 1998. *A History of Mathematics: an Introduction*. 2nd edition. New York: Harper Collins.

One of the best and most reliable general histories of mathematics available. It is especially strong in its treatment of non-European mathematical developments.

———. 1995. Ideas of calculus in Islam and India. *Math. Magazine* **68**:163–174.

This looks at the discovery in eleventh century Egypt of the formula for sums of integer powers and the discovery of the series expansions of the sine and cosine in sixteenth century India.

Roy, R. 1990. The discovery of the series formula for π by Leibniz, Gregory and Nīlakaṇṭha. *Math. Magazine*. **63**:291–306.

An account of three discoveries of the series expansion for the arc tangent function. A very accessible description of Nīlakaṇṭha's derivation of this series.

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